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Classification of Quasi-Sum Production Functions with Allen Determinants

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Abstract. In this paper, we derive an explicit formula for the determinant of Allen's matrix of quasi-sum production functions. We completely classify the quasi-sum production functions by using their Allen determinants. Further, we give some geometric applications of Allen determinants.

1. Introduction

In economics, a *production function* is a mathematical expression which denotes the physical relations between the output generated of a firm, an industry or an economy and inputs that have been used. Explicitly, a production function is a map which has non-vanishing first derivatives defined by

$$f : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+, \quad f = f(x_1, x_2, ..., x_n), \tag{1.1}$$

where f is the quantity of output, n is the number of inputs and $x_1, x_2, ..., x_n$ are the inputs. For more detailed properties of production functions, see [4, 13, 17, 19, 20].

A production function f is called *quasi-sum* if there are continuous strict monotone functions $h_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $i = 1, ..., n$, and there exist an interval $I \subset \mathbb{R}$ of positive length and a continuous strict monotone function $F : I \longrightarrow \mathbb{R}$ such that for each $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}_+^n$ we have $h_1(x_1) + ... + h(x_n) \in I$ and

$$f(\mathbf{x}) = F(h_1(x_1) + ... + h(x_n)). \tag{1.2}$$

The justification for studying production functions of quasi-sum form is that these functions appear as solutions of the general bisymmetry equation and they are related to the problem of consistent aggregation [1, 7]. A quasi-sum production function is called *quasi-linear* if at most one of $F, h_1, ..., h_n$ from (1.2) is a nonlinear function.

The most common quantitative indices of production factor substitutability are forms of the elasticity of substitution. R.G.D. Allen and J.R. Hicks [2] suggested two generalizations of Hicks' original two variable elasticity concept.

The first concept, called Hicks elasticity of substitution, is defined as follows.

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Let $f(x_1, ..., x_n)$ be a production function. Then *Hicks elasticity of substitution* of the $i-$th production variable with respect to the $j-$th production variable is given by

$$H_{ij}(\mathbf{x}) = -\frac{\frac{1}{x_i f_i} + \frac{1}{x_j f_j}}{\frac{f_{ii}}{(f_i)^2} - \frac{2 f_{ij}}{f_i f_j} + \frac{f_{jj}}{(f_j)^2}} \quad (\mathbf{x} \in \mathbb{R}^n_+,\ i, j = 1, ..., n,\ i \neq j), \tag{1.3}$$

where $f_i = \partial f / \partial x_i$, $f_{ij} = \partial^2 f / \partial x_i \partial x_j$.

L. Losonczi [14] classified homogeneous production functions of 2 variables, having constant Hicks elasticiy of substitution. Then, the classification of L. Losonczi was extended to n variables by B-Y. Chen [6].

The second concept, investigated by R.G.D. Allen [2] and H. Uzawa [18] , is the following:

Let f be a production function. Then *Allen elasticity of substitution* of the $i-$th production variable with respect to the $j-$th production variable is defined by

$$A_{ij}(\mathbf{x}) = -\frac{x_1 f_1 + x_2 f_2 + ... + x_n f_n}{x_i x_j} \frac{D_{ij}}{D} \quad (\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n_+,\ i, j = 1, ..., n,\ i \neq j), \tag{1.4}$$

where D is the determinant of the matrix

$$M(f) = \begin{pmatrix} 0 & f_1 & \cdots & f_{n-1} & f_n \\ f_1 & f_{11} & \cdots & f_{1n-1} & f_{1n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ f_{n-1} & f_{1n-1} & \cdots & f_{n-1n-1} & f_{n-1n} \\ f_n & f_{1n} & \cdots & f_{n-1n} & f_{nn} \end{pmatrix} \tag{1.5}$$

and D_{ij} is the co-factor of the element f_{ij} in the determinant D ($D \neq 0$ is assumed). $M(f)$ is called the *Allen's matrix* and we call $\det(M(f))$ the *Allen determinant*.

It is a simple calculation to show that in case of two variables Hicks elasticity of substitution coincides with Allen elasticity of substitution.

In this paper, we focus on the singularity of the Allen's matrix of quasi-sum production functions, by analogy to the Hessian Determinant Formula given by B.-Y. Chen in [10] for the composite functions of the form:

$$f(\mathbf{x}) = F(h_1(x_1) + h_2(x_2) + ... + h_n(x_n)). \tag{1.6}$$

We give an explicit formula for Allen determinant of quasi-sum production functions. We classify the quasi-sum production functions by their Allen determinants. Further we give some geometric applications of Allen determinants.

2. Quasi-Sum Production Functions with Allen Determinants

Allen Determinant Formula. *Let $f = F(h_1(x_1) + h_2(x_2) + ... + h_n(x_n))$ be a quasi-sum production function of n variables. Then the determinant of the Allen's matrix $M(f)$ is given by*

$$\det(M(f)) = -(F')^{n+1} \sum_{j=1}^{n} h''_1 ... h''_{j-1} \left(h'_j\right)^2 h''_{j+1} ... h''_n, \tag{2.1}$$

where $h'_j = \frac{dh_j}{dx_j}$, $h''_j = \frac{d^2 h_j}{dx_j^2}$, for $j = 1, ..., n$, and $F' = F'(u)$, for $u = h_1(x_1) + h_2(x_2) + ... + h_n(x_n)$.

Proof. Let $f = F(h_1(x_1) + h_2(x_2) + ... + h_n(x_n))$ be a twice differentiable quasi-sum production function. First we show that the equality (2.1) is satisfied for $n = 2$. For this, the Allen's matrix for the two variables quasi-sum production function $f = F(h_1(x_1) + h_2(x_2))$ is given by

$$M(f) = \begin{pmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{pmatrix}, \tag{2.2}$$

where $f_i = \frac{\partial f}{\partial x_i}$, $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Then, in our case,

$$f_i = h_i' F', \; f_{ij} = h_i' h_j' F'', \text{ for } i \neq j, \text{ and } f_{ii} = \left(h_i'\right)^2 F'' + h_i'' F'. \tag{2.3}$$

So, we write the Allen's matrix for the two variables quasi-sum production function
$f = F(h_1(x_1) + h_2(x_2))$ as follows:

$$M(f) = \begin{pmatrix} 0 & h_1'F' & h_2'F' \\ h_1'F' & \left(h_1'\right)^2 F'' + h_1''F' & h_1'h_2'F'' \\ h_2'F' & h_1'h_2'F'' & \left(h_2'\right)^2 F'' + h_2''F' \end{pmatrix}.$$

The determinat of $M(f)$ is

$$\det(M(f)) = -(F')^3 \left[\left(h_1'\right)^2 \left(h_2''\right) + \left(h_1''\right)\left(h_2'\right)^2\right]. \tag{2.4}$$

Therefore, we have formula (2.1) for $n = 2$.

Now, we prove the formula (2.1) by mathematical induction. Let us assume that (2.1) holds for $n = k$, with $n \geq 2$. We shall show that (2.1) is satisfied for $n = k + 1$. For $n = k + 1$, we have

$$M(f) = \begin{pmatrix} 0 & f_1 & \dots & f_k & f_{k+1} \\ f_1 & f_{11} & \dots & f_{1k} & f_{1k+1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ f_k & f_{1k} & \dots & f_{kk} & f_{kk+1} \\ f_{k+1} & f_{1k+1} & \dots & f_{kk+1} & f_{k+1k+1} \end{pmatrix}, \tag{2.5}$$

and then the determinant of $M(f)$ is given by

$$\det(M(f)) = \begin{vmatrix} 0 & h_1'F' & \dots & h_k'F' & h_{k+1}'F' \\ h_1'F' & \left(h_1'\right)^2 F'' + h_1''F' & \dots & h_1'h_k'F'' & h_1'h_{k+1}'F'' \\ \vdots & \vdots & \dots & \vdots & \vdots \\ h_k'F' & h_1'h_k'F'' & \dots & \left(h_k'\right)^2 F'' + h_k''F' & h_k'h_{k+1}'F'' \\ h_{k+1}'F' & h_1'h_{k+1}'F'' & \dots & h_k'h_{k+1}'F'' & \left(h_{k+1}'\right)^2 F'' + h_{k+1}''F' \end{vmatrix}. \tag{2.6}$$

Now, we apply some elementary transformations for the determinant.

We replace the $(k+2)$ −th row by $(k+2)$ −th row minus h_{k+1}'/h_k' times the $(k+1)$ −th row; then we obtain

$$\det(M(f)) = \begin{vmatrix} 0 & h_1'F' & \dots & h_k'F' & h_{k+1}'F' \\ h_1'F' & \left(h_1'\right)^2 F'' + h_1''F' & \dots & h_1'h_k'F'' & h_1'h_{k+1}'F'' \\ \vdots & \vdots & \dots & \vdots & \vdots \\ h_k'F' & h_1'h_k'F'' & \dots & \left(h_k'\right)^2 F'' + h_k''F' & h_k'h_{k+1}'F'' \\ 0 & 0 & \dots & -\frac{h_{k+1}'h_k''}{h_k'}F' & h_{k+1}''F' \end{vmatrix}$$

$$= \frac{h'_{k+1}h''_k}{h'_k}F'\begin{vmatrix} 0 & h'_1F' & \dots & h'_{k-1}F' & h'_{k+1}F' \\ h'_1F' & \left(h'_1\right)^2F''+h''_1F' & \dots & h'_1h'_{k-1}F'' & h'_1h'_{k+1}F'' \\ \vdots & \vdots & \dots & \vdots & \vdots \\ h'_{k-1}F' & h'_1h'_{k-1}F'' & \dots & \left(h'_{k-1}\right)^2F''+h''_{k-1}F' & h'_{k-1}h'_{k+1}F'' \\ h'_kF' & h'_1h'_kF'' & \dots & h'_{k-1}h'_kF'' & h'_kh'_{k+1}F'' \end{vmatrix}$$

$$+h''_{k+1}F'\underbrace{\begin{vmatrix} 0 & h'_1F' & \dots & h'_kF' & h'_kF' \\ h'_1F' & \left(h'_1\right)^2F''+h''_1F' & \dots & h'_1h'_kF'' & h'_1h'_kF'' \\ \vdots & \vdots & \dots & \vdots & \vdots \\ h'_{k-1}F' & h'_1h'_{k-1}F'' & \dots & \left(h'_{k-1}\right)^2F''+h''_{k-1}F' & h'_{k-1}h'_kF'' \\ h'_kF' & h'_1h'_kF'' & \dots & h'_{k-1}h'_kF'' & \left(h'_k\right)^2F''+h''_kF' \end{vmatrix}}_{-(F')^k\sum_{j=1}^k h''_1\dots h''_{j-1}\left(h'_j\right)^2h''_{j+1}\dots h''_k}$$

$$= h''_{k+1}\left(-(F')^{k+1}\sum_{j=1}^{k}h''_1\dots h''_{j-1}\left(h'_j\right)^2h''_{j+1}\dots h''_k\right) \tag{2.7}$$

$$+\frac{h'_{k+1}h''_k}{h'_k}F'\begin{vmatrix} 0 & h'_1F' & \dots & h'_{k-1}F' & h'_{k+1}F' \\ h'_1F' & \left(h'_1\right)^2F''+h''_1F' & \dots & h'_1h'_{k-1}F'' & h'_1h'_{k+1}F'' \\ \vdots & \vdots & \dots & \vdots & \vdots \\ h'_{k-1}F' & h'_1h'_{k-1}F'' & \dots & \left(h'_{k-1}\right)^2F''+h''_{k-1}F' & h'_{k-1}h'_{k+1}F'' \\ h'_kF' & h'_1h'_kF'' & \dots & h'_{k-1}h'_kF'' & h'_kh'_{k+1}F'' \end{vmatrix}.$$

The determinant from the formula (2.7) can be calculated by replacing the $(k+1)$ −th row by the $(k+1)$ −th row minus h'_k/h'_{k-1} times k−th row,

$$|B| \overset{not}{=} \begin{vmatrix} 0 & h'_1F' & \dots & h'_{k-1}F' & h'_{k+1}F' \\ h'_1F' & \left(h'_1\right)^2F''+h''_1F' & \dots & h'_1h'_{k-1}F'' & h'_1h'_{k+1}F'' \\ \vdots & \vdots & \dots & \vdots & \vdots \\ h'_{k-1}F' & h'_1h'_{k-1}F'' & \dots & \left(h'_{k-1}\right)^2F''+h''_{k-1}F' & h'_{k-1}h'_{k+1}F'' \\ h'_kF' & h'_1h'_kF'' & \dots & h'_{k-1}h'_kF'' & h'_kh'_{k+1}F'' \end{vmatrix}$$

$$= \frac{h'_{k+1}h''_k}{h'_k}F'\begin{vmatrix} 0 & h'_1F' & \dots & h'_{k-1}F' & h'_{k+1}F' \\ h'_1F' & \left(h'_1\right)^2F''+h''_1F' & \dots & h'_1h'_{k-1}F'' & h'_1h'_{k+1}F'' \\ \vdots & \vdots & \dots & \vdots & \vdots \\ h'_{k-1}F' & h'_1h'_{k-1}F'' & \dots & \left(h'_{k-1}\right)^2F''+h''_{k-1}F' & h'_{k-1}h'_{k+1}F'' \\ 0 & 0 & \dots & -\frac{h'_kh''_{k-1}F'}{h'_{k-1}} & 0 \end{vmatrix}$$

$$= \frac{h'_{k+1}h''_k}{h'_k}\frac{h'_k h''_{k-1}}{h'_{k-1}}(F')^2$$

$$\cdot\begin{vmatrix} 0 & h'_1F' & \cdots & h'_{k-1}F' & h'_{k+1}F' \\ h'_1F' & \left(h'_1\right)^2F''+h''_1F' & \cdots & h'_1h'_{k-1}F'' & h'_1h'_{k+1}F'' \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ h'_{k-2}F' & h'_1h'_{k-2}F'' & \cdots & \left(h'_{k-2}\right)^2F''+h''_{k-2}F' & h'_{k-2}h'_{k+1}F'' \\ h'_{k-1}F' & h'_1h'_{k-1}F'' & \cdots & h'_{k-2}h'_{k-1}F'' & h'_{k-1}h'_{k+1}F'' \end{vmatrix}.$$

By similar calculations, we finally obtain:

$$|B| = \frac{h'_{k+1}h''_k}{h'_k}\frac{h'_k h''_{k-1}}{h'_{k-1}}\frac{h'_{k-1}h''_{k-2}}{h'_{k-2}}\cdots\frac{h'_4h''_3}{h'_3}\frac{h'_3h''_2}{h'_2}(F')^{k-2}\cdot\begin{vmatrix} 0 & h'_1F' & h'_{k+1}F' \\ h'_1F' & \left(h'_1\right)^2F''+h''_1F' & h'_1h'_{k+1}F'' \\ 0 & -\frac{h'_2h''_1}{h'_1}F' & 0 \end{vmatrix} \tag{2.8}$$

$$= \frac{h'_{k+1}h''_k}{h'_k}\frac{h'_k h''_{k-1}}{h'_{k-1}}\frac{h'_{k-1}h''_{k-2}}{h'_{k-2}}\cdots\frac{h'_4h''_3}{h'_3}\frac{h'_3h''_2}{h'_2}(F')^{k-2}\frac{h'_2h''_1}{h'_1}F'\left(-h'_1h'_{k+1}(F')^2\right)$$

$$= -h''_1h''_2\ldots h''_k\left(h'_{k+1}\right)^2(F')^{k+1}.$$

From (2.7) and (2.8), we get:

$$\det(M(f)) = -h''_{k+1}\left((F')^{k+1}\sum_{j=1}^{k}h''_1\ldots h''_{j-1}\left(h'_j\right)^2h''_{j+1}\ldots h''_k\right) - h''_1h''_2\ldots h''_k\left(h'_{k+1}\right)^2(F')^{k+1}$$

$$= -(F')^{k+1}\sum_{j=1}^{k+1}h''_1\ldots h''_{j-1}\left(h'_j\right)^2h''_{j+1}\ldots h''_n,$$

which completes the proof.

F being strict monotone, $F' \neq 0$; then, by the Allen Determinant Formula, it follows immediately

Corollary 2.1. *The singularity of Allen's matrix for quasi-sum production function $f = F(h_1(x_1) + h_2(x_2) + \ldots + h_n(x_n))$ depends only on the functions $h_1, \ldots, h_n$.*

Next result completely classifies quasi-sum production functions whose Allen's matrices are singular.

Theorem 2.2. *Let f be a twice differentiable quasi-sum production function, $f(\mathbf{x}) = F(h_1(x_1) + h_2(x_2) + \ldots + h_n(x_n))$. Then the Allen's matrix $M(f)$ is singular if and only if f is one of the following forms:*

(1) $f = F(c_1x_1 + c_2x_2 + h_3(x_3) + \ldots + h_n(x_n))$, *where c_1, c_2 are nonzero constants and $F, h_3, \ldots, h_n$ are strict monotone functions;*

(2) $f = F\left(\sum_{i=1}^{n} c_i \ln|x_i + d_i| + e_i\right)$, *where c_i are nonzero constans and d_i, e_i are some constants.*

Proof. Let us consider the twice differentiable quasi-sum production function given by

$$f = F(h_1(x_1) + h_2(x_2) + \ldots + h_n(x_n)). \tag{2.9}$$

Hence $F, h_1, \ldots, h_n$ are continuous strict monotone functions. Applying the Allen Determinant Formula for (2.9), we get

$$\det(M(f)) = -(F')^{n+1}\sum_{j=1}^{n}h''_1\ldots h''_{j-1}\left(h'_j\right)^2h''_{j+1}\ldots h''_n. \tag{2.10}$$

If the Allen's matrix $M(f)$ is singular, then by (2.10) it follows that:

$$\sum_{j=1}^{n} h_1''...h_{j-1}''\left(h_j'\right)^2 h_{j+1}''...h_n'' = 0. \tag{2.11}$$

Case 1. One of the $h_1'',...,h_n''$ vanishes. We may assume that $h_1'' = 0$. Hence we get from (2.11)

$$h_2''...h_n'' = 0, \tag{2.12}$$

because $h_1' \neq 0$, h_1 being strict monotone. From (2.12), we may assume $h_2'' = 0$ and thus we have for $k = 1, 2$

$$h_k(x_k) = c_k x_k + d_k,$$

where c_k are nonzero constants and d_k are some constants. Therefore the production function takes the form:

$$f = F(c_1x_1 + c_2x_2 + h_3(x_3) + ... + h_n(x_n)),$$

where c_1, c_2 are nonzero constants and $h_3, ..., h_n$ are strict monotone functions. This gives first part of the theorem.

Case 2. $h_1'',...,h_n''$ are nonzero. Then from (2.11), by dividing with the product $h_1'',...,h_n''$, we write

$$\frac{\left(h_1'\right)^2(x_1)}{h_1''(x_1)} + ... + \frac{\left(h_n'\right)^2(x_n)}{h_n''(x_n)} = 0. \tag{2. 13}$$

Taking partial derivative of (2.13) with respect to x_i, we derive

$$\frac{h_i'h_i'''}{\left(h_i''\right)^2} = 2,\ i = 1, ..., n. \tag{2.14}$$

By solving (2.14) we find $h_i(x_i) = c_i \ln|x_i + d_i| + e_i$, where c_i are nonzero constans and d_i, e_i are some constants. If f has the form (1) or (2), it is easily seen that $M(f)$ is singular. This completes the proof.

3. An Application of Allen Determinants for Composite Functions

Theorem 3.1. *Let $F(u)$ be a twice differentiable function with $F'(u) \neq 0$ and let*

$$f = F\left(\sum_{i=1}^{n} b_i x_i^{c_i}\right)$$

be the composite of F and $r(\mathbf{x}) = \sum_{i=1}^{n} b_i x_i^{c_i}$, where b_i $(i = 1, ..., n)$ are nonzero constants. Then the Allen's matrix $M(f)$ of f is singular if and only if one of the following conditions occurs:

(*i*) *At least one of the $c_1, ..., c_n$ vanishes;*

(*ii*) *At least two of $c_1, ..., c_n$ are equal to one.*

Proof. Under the hypothesis of theorem, we can write

$$h_i(x_i) = b_i x_i^{c_i},\ i = 1, ..., n. \tag{3.1}$$

Thus the composite function $f = F\left(\sum_{i=1}^{n} b_i x_i^{c_i}\right)$ takes the form:

$$f = F(h_1(x_1) + ... + h_n(x_n)).$$

From (3.1) , we have

$$h_i' = b_i c_i x_i^{c_i - 1} \text{ and } h_i'' = b_i c_i (c_i - 1) x_i^{c_i - 2}, \ i = 1, ..., n.$$

If we apply the Allen Determinant Formula for the composite function $f = F \circ r$, then we get

$$\begin{aligned} \det(M(f)) &= -(F')^{n+1} \\ &\cdot \sum_{j=1}^{n} \Big\{ \left[b_1 c_1 (c_1 - 1) x_1^{c_1 - 2} \right] ... \left[b_j c_j \left(c_j - 1\right) x_j^{c_j - 2} \right] \left[b_j c_j x_j^{c_j - 1} \right]^2 \\ &\cdot \left[b_{j+1} c_{j+1} \left(c_{j+1} - 1\right) x_{j+1}^{c_{j+1} - 2} \right] ... \left[b_n c_n (c_n - 1) x_n^{c_n - 2} \right] \Big\} \\ &= -(F')^{n+1} \left(\prod_{k=1}^{n} b_k c_k x_k^{c_k - 2} \right) \left(\sum_{j=1}^{n} b_j (c_1 - 1) ... \left(c_{j-1} - 1\right) c_j (c_{1+1} - 1) ... (c_n - 1) x_j^{c_j} \right). \end{aligned}$$

From the last equality, if $M(f)$ is a singular matrix then either at least one of the $c_1, ..., c_n$ is equal to zero or at least two of the $c_1, ..., c_n$ are equal to one.

The converse is easily verified.

4. Geometric Interpretations of Allen Determinants

Let M^n be a hypersurface of a Euclidean space $\mathbb{E}^{n+1}$. For general references on the geometry of hypersurfaces see [3, 4] .

The *Gauss map* $\nu : M^n \longrightarrow S^{n+1}$ maps M^n to the unit hypersphere S^n of $\mathbb{E}^{n+1}$. The differential $d\nu$ of the Gauss map ν is known as the *shape operator* or Weingarten map. Denote by T_pM^n the tangent space of M^n at the point $p \in M^n$. Then, for $v, w \in T_pM^n$, the shape operator A_p at the point $p \in M^n$ is defined by

$$g\left(A_p(v), w\right) = g(d\nu(v), w),$$

where g is the induced metric tensor on M^n from the Euclidean metric on $\mathbb{E}^{n+1}$.

The determinant of the shape operator A_p is called the *Gauss-Kronocker* curvature.

Let (N, g) be a Riemannian manifold. For more detailed properties of geometric structures on Riemannian manifolds, see [12] . A *Riemannian connection*, also called Levi-Civita connection, on the Riemannian manifold (N, g) is an affine connection which is compatible with metric, i.e, $\nabla g = 0$ and symmetric, i.e, $\nabla_X Y - \nabla_Y X = [X, Y]$, for any vector fields X and Y on N, where $[,]$ is the Lie bracket.

The *Riemannian curvature tensor* R is given in terms of ∇ by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

A Riemannian manifold is called a *flat space* if its Riemannian curvature tensor vanishes identically.

Let σ be a two dimensional subspace of the tangent space T_pN and let $u, v \in \sigma$ be two linearly independent vectors such that $\sigma = Sp(u, v)$. Then the *sectional curvature* of σ at the point $p \in N$ is a real number defined by

$$K(u, v) = K(\sigma) = \frac{g(R(u, v)v, u)}{g(u, u)g(v, v) - g(u, v)^2}.$$

The *Ricci tensor* of a Riemannian manifold N at a point $p \in N$ is defined to be the trace of the linear map $T_pN \longrightarrow T_pN$ given by

$$w \longmapsto R(w, u)v.$$

A Riemannian manifold is called *Ricci-flat* if its Ricci tensor vanishes identically.

The following result is well-known from [4, 7] .

Proposition 4.1. *For the production hypersurface of* $\mathbb{E}^{n+1}$ *defined by*

$$L(\mathbf{x}) = (x_1, ..., x_n, f(x_1, ..., x_n)),$$

we have:

(i) The Gauss-Kronecker curvature G is

$$G = \frac{\det\left(f_{ij}\right)}{w^{n+2}},$$

with $w = \sqrt{1 + \sum_{i=1}^{n} f_i^2}$.

(ii) The sectional curvature K_{ij} *of the plane section spanned by* $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$ *is given by*

$$K_{ij} = \frac{f_{ii} f_{jj} - f_{ij}^2}{w^2\left(1 + f_i^2 + f_j^2\right)}.$$

(iii) The Riemannian curvature tensor R and the metric tensor g satisfy

$$g\left(R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) = \frac{f_{il} f_{jk} - f_{ik} f_{jl}}{w^4}. \tag{4.1}$$

Corollary 4.2. *The Allen's matrix of a twice differentiable quasi-sum production function with more than two variables is singular if its production hypersurface is a flat space.*

Proof. Let $f = F(h_1(x_1) + ... + h_n(x_n))$ be a twice differentiable quasi-sum production function, $n \geq 3$. If the production hypersurface of f is a flat space then, by Theorem 4.1 of [7], f is quasi-linear, namely at most one of $F, h_1, ..., h_n$ is a nonlinear function. This is a special case of statement (i) of Theorem 2.2. Therefore, we obtain that the Allen's matrix of f is singular.

Corollary 4.3. *Let* $f = F(h_1(x_1) + ... + h_n(x_n))$ *be a twice differentiable quasi-sum production function with one of the* $h_1, ..., h_n$ *linear function and* $F'' \neq 0$. *Then the production hypersurface of f has vanishing Gauss-Kronocker curvature if and only if the Allen's matrix of f is singular.*

Proof. Let us assume that the production hypersurface of f has vanishing Gauss-Kronocker curvature. Then, by Theorem 5.1 of [7], f is of the form:

$$f = F\left(c_1 x_1 + c_2 x_2 + \sum_{i=3}^{n} h_i(x_i)\right), \tag{4.2}$$

where c_1, c_2 are nonzero constants and $F, h_3, ..., h_n$ are strict monotone functions. It means from Theorem 2.2 that the Allen's matrix of f is singular.

Conversely, under the hypothesis of the corollary, if the Allen's matrix of f is singular, then, by Theorem 2.2, f is of the form (4.2). Therefore, by Theorem 5.1 of [7], we obtain that the production hypersurface of f has vanishing Gauss-Kronocker curvature.

Theorem 4.4. *Let* $f(x_1, x_2, x_3)$ *be a twice differentiable production function of 3 variables. If the production hypersurface of f in* $\mathbb{E}^4$ *is a flat space then its Allen's matrix* $M(f)$ *is singular.*

Proof. The Allen's matrix for the production function $f(x_1, x_2, x_3)$ is given by

$$M(f) = \begin{pmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{pmatrix}. \tag{4.3}$$

For the above matrix we find

$$\det(M(f)) = -f_1^2\left(f_{22}f_{33}-f_{23}^2\right) - f_2^2\left(f_{11}f_{33}-f_{13}^2\right) - f_3^2\left(f_{11}f_{22}-f_{12}^2\right) \tag{4.4}$$

$$+2f_1f_2\left(f_{12}f_{33}-f_{13}f_{23}\right) - 2f_1f_3\left(f_{12}f_{23}-f_{13}f_{22}\right) + 2f_2f_3\left(f_{11}f_{23}-f_{13}f_{12}\right).$$

On the other hand, if the production hypersurface of f is a flat space, then, by the statement (iii) of Proposition 4.1, we have

$$f_{kn}f_{ml} = f_{kl}f_{mn}, \tag{4.5}$$

i.e. all brackets in the formula (4.4) are zero. From (4.4) and (4.5) we obtain $\det(M(f)) = 0$, which completes the proof.

Remark 4.5. The Theorem 4.4 also holds in case the production hypersurface of f in $\mathbb{E}^4$ is a Ricci-flat space or has vanishing sectional curvature function.

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